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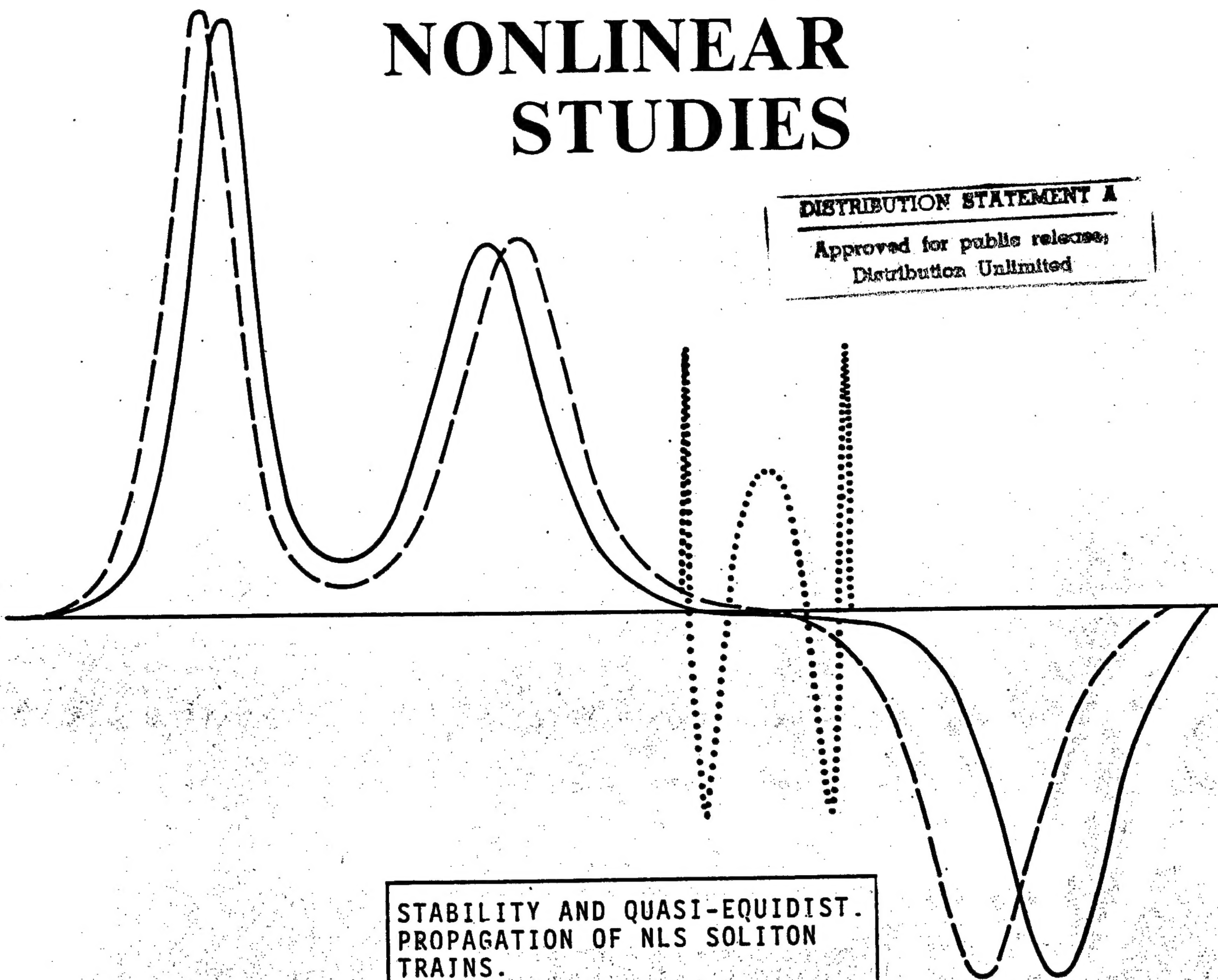

STABILITY AND QUASI-EQUIDIST. PROPAGATION OF NLS SOLITON TRAINS.

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Stability and Quasi-Equidistant Propagation of NLS Soliton Trains.

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Using the complex Toda chain (CTC) as a model for the propagation of the N-soliton pulse trains of the nonlinear Schrödinger (NLS) equation, we can predict the stability and the asymptotic behavior of these trains. We show that the following asymptotic regimes are stable: (i) asymptotically free propagation of all N solitons; (ii) bound state regime where the N solitons move quasi-equidistantly; and (iii) various different combinations of (i) and (ii). On the example of $N = 3$ we show how the CTC model can be used to determine analytically the set of initial soliton parameters corresponding to regime (ii). We compare these analytical results against the corresponding numerical solutions of the NLS and find excellent agreement in most cases. We concentrate on the quasi-equidistant propagation of all N solitons because it is of importance for optical fiber soliton communication. We check numerically that such propagation takes place for $N = 2$ to 8. Finally we propose realistic configurations for the sets of the amplitudes, for which the trains show quasi-equidistant behavior to very large run lengths.

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I. INTRODUCTION

One of the important problems in optical fiber soliton communication is to achieve as high of a bit rate as possible. In order to do this, one needs to be able to pack the solitons into as short of a space as possible. However, if the solitons are too close together, then their mutual (linear and nonlinear) interactions can cause them to collide and/or separate, thereby corrupting the signal. The current solution of this problem is simply to require each soliton to be sufficiently far apart from all others (usually 6 or so soliton widths) so that such interactions can be totally neglected. However, at the same time, it was predicted [1,2] and experimentally confirmed [3] that for certain values of relative soliton parameters, this separation can be reduced, and at the same time, still maintain signal integrity. Our main purpose here is to analytically and numerically detail the soliton parameter regime, inside of which, signal integrity can be maintained. In particular, we are interested to determine how one may use this inter-soliton interaction for *stabilizing* a soliton train.

Any communication signal will be composed of "random" combinations of 0's and 1's. It can also be viewed as being composed of a random collection of N-soliton trains, with varying widths of 0's between them. Thus it is then adequate for us to simply analyze the stability of individual N-soliton trains, for finite N. This we will do and will study a nonperiodic and finite train (chain) of soliton pulses by both analytical and numerical methods using and developing the ideas in [4,5].

The basic model and description of N-soliton trains in optical fibers is provided by the nonlinear Schrödinger (NLS) equation and its perturbed version:

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u(x,t) = iR[u]. \tag{1}$$

This equation describes a variety of wave interactions, including solitons in nonlinear fiber optics [1,6–11] and spatial solitons in nonlinear refractive media [9].

The inverse scattering method [6] allows one to solve exactly Eq. (1) when $R[u]=0$ and to calculate explicitly its N-soliton solutions. However for our purposes, this method is impractical for two reasons. First, there are important perturbations of this system which have no explicit solutions. Second, an approximate method can serve much better than an exact approach since the N-soliton trains here are rather special and can only be approximated by N-soliton solutions. Such trains are actually sums of 1-soliton pulses, which are spaced almost equally apart, have almost equal amplitudes, and move with essentially the same velocity. More specifically, they are the solutions to Eq. (1) with $R[u]=0$ satisfying the following initial conditions:

$$u(x,0) = \sum_{k=1}^{N} \frac{2\nu_k e^{i\phi_{k,0}(x)}}{\cosh(2\nu_k(x-\xi_{k,0})}, \tag{2}$$

where $\phi_{k,0}(x) = 2\mu_0(x-\xi_{k,0}) + \delta_{k,0}$, and $\delta_{k,0}$, $\xi_{k,0}$, $2\nu_{k,0}$, and $2\mu_{k,0}$ are the initial phase, position, amplitude and velocity respectively of the k-th soliton; by $2\nu_0$ and $2\mu_0$ we mean the averages of the initial amplitudes and velocities.

An effective method for studying the interaction of such trains of soliton pulses was first proposed by Karpman and Solov'ev (KS), for the simplest nontrivial case of a 2-soliton interaction [12]. For further developments and analysis for different physically important perturbations can be found in [13,14] and the references therein; for a review see Ref. [7]. The KS method is based on the adiabatic approximation. It is valid for any collection of well separated solitons, such that their mutual interactions will lead to a slow deformation in the soliton parameters. As a result the soliton interactions should be described by a dynamical system for ξ_t, ν_t, μ_t and δ_t.

Gorshkov [15] and Arnold [16], have conjectured that an infinite train of out-of-phase soliton pulses, with equal amplitudes and velocities could be described by the real Toda chain:

$$\frac{d^2 q_k}{d\tau^2} = e^{q_{k+1}-q_k} - e^{q_k - q_{k-1}}, \tag{3}$$

where $\tau = 4\nu_0 t$, $k = 0, \pm 1, \pm 2, \ldots$ and q_k are real functions related to the soliton positions. This system will be referred to as the real Toda chain (RTC). The validity of this description for finite number of solitons (see also below) has been verified also numerically [17].

Recently in Refs. [4,5,17], the Karpman–Solov'ev method was extended to N-soliton pulses, and then with additional approximations, was reduced to the complex Toda chain equations (CTC) (3) with N sites. The corresponding system of equations is (3), but with $k = 1, \ldots, N$ and $e^{-q_0} \equiv e^{q_{N+1}} \equiv 0$. The complex valued functions $q_k(t)$ are related to the soliton parameters by:

$$\begin{aligned} q_{k+1} - q_k = &-2\nu_0(\xi_{k+1}-\xi_k) + \ln 4\nu_0^2 \\ &+ i\left(\pi + 2\mu_0(\xi_{k+1}-\xi_k) - (\delta_{k+1}-\delta_k)\right). \end{aligned} \tag{4}$$

Thus the problem of determining the evolution of an NLS N-soliton train has been reduced to the problem of solving the CTC for N sites. CTC, like RTC is integrable and we may use the special techniques valid for integrable lattices, see [18–20].

Our main results are the following:

(i) we show by analyzing the exact analytic solutions of CTC, that it has several qualitatively different classes of asymptotic regimes. Besides the asymptotically free motion (which is the only one possible for the RTC [19,20]), CTC allows also for: (a) bound state regime when all the N particles move quasi-equidistantly; (b) all possible intermediate regimes when one (or several) group(s) of particles form bound state(s) and the rest of them go into free motion asymptotics. In addition to these relatively stable regimes of motion there are also less stable regions in the space of soliton parameters, where one regime switches into another one. There one can find

(c) singular solutions, and (d) various types of degenerate solutions.

(ii) we compare the predictions from the CTC model with the numerical solutions of the NLS, that bound states regimes indeed take place in the soliton interactions and are very well described by the CTC model. Our analytic approach allows us to predict the set of initial parameters, for which these asymptotic regimes takes place. We put special stress on the bound state and quasi-equidistant regimes since such solutions would be the desirable in long range fiber optics communications.

II. ASYMPTOTIC REGIMES OF THE CTC

As in [4,5], one can generalize the RTC [18–20] to the complex CTC case. We list the four most important points concerning this below:

S1) The CTC Lax representation is the same as for the RTC: $\dot{L} = [B, L]$, where

$$L = \sum_{k=1}^{N} \left(b_k E_{kk} + a_k (E_{k,k+1} + E_{k+1,k}) \right), \qquad (5)$$

Here $a_k = \frac{1}{2} e^{(q_{k+1}-q_k)/2}$ and $b_k = \frac{1}{2}(\mu_k + i\nu_k)$. The matrices $(E_{kn})_{pq} = \delta_{kp}\delta_{nq}$, and $(E_{kn})_{pq} = 0$ whenever p or q becomes 0 or $N+1$.

S2) The integrals of motion in involution are provided by the eigenvalues, ζ_k, of $L_0 = L(\tau = 0)$.

S3) The solutions of both the CTC and the RTC are determined by the scattering data S_{L_0} of L_0. If $\zeta_k \neq \zeta_j$ for $k \neq j$, then $S_{L_0} = \{\zeta_k, r_k\}_{k=1}^{N}$. Here r_k are the first components of the corresponding eigenvectors $\xi^{(k)}$ of L_0, which are fixed (up to an overall sign) by the condition $\sum_{s=1}^{N} \left(\xi_s^{(k)}\right)^2 = 1$, [see [19,20]]. S_{L_0} uniquely determines both L_0 and the solution of the CTC.

S4) Lastly, ζ_k uniquely determine the asymptotic behavior of the solutions of the CTC and can be calculated from the initial conditions. We will use this fact to describe the different classes of asymptotic behavior.

In addition to the dynamical variables becoming complex valued, there are other, important differences between the RTC and CTC, and their properties. For the RTC, one has that [19,20], both the eigenvalues, ζ_k, and the coefficients, r_k, are always real-valued. Moreover, one can prove that $\zeta_k \neq \zeta_j$ for $k \neq j$. As a consequence the only possible asymptotic behavior in the RTC is an asymptotically separating, free motion of the solitons.

For CTC not only the dynamical variables q_k, but also $\zeta_k = \kappa_k + i\eta_k$ and r_k take complex values. The collection of eigenvalues, ζ_k, still determines the asymptotic behavior of the solitons. In particular, it is κ_k that determines the asymptotic velocity of the k-th soliton. For simplicity and without loss of generality we assume that: $\operatorname{tr} L_0 = 0$; $\zeta_k \neq \zeta_j$ for $k \neq j$ (this does not necessarily mean that $\kappa_k \neq \kappa_j$); and that the κ_k's are ordered as: $\kappa_1 \leq \kappa_2 \leq \ldots \leq \kappa_N$. Then in any train of solitons, there are three possible general configurations:

D1) $\kappa_k \neq \kappa_j$ for $k \neq j$, i.e. the asymptotic velocities are all different. Then we have asymptotically separating, free solitons, see [4,5]

D2) $\kappa_1 = \kappa_2 = \ldots = \kappa_N$, i.e. all N solitons will move with the same mean asymptotic velocity, and therefore will form a "bound state". The key question now will be the nature of the internal motions in such a bound state. In particular, one would want any two adjacent solitons to move quasi-equidistantly.

D3) One may have also a variety of intermediate situations when only one group (or several groups) of particles move with the same mean asymptotic velocity; then they would form one (or several) bound state(s) and the rest of the particles will have free asymptotic motion.

Obviously the cases D2) and D3) have no analogies in the RTC and physically are qualitatively different from D1). The same is also true for the special degenerate cases, where two or more of the ζ_k's may become equal. These cases will be considered elsewhere. Another type of solutions of the CTC which should be dealt with separately are the singular solutions, see e.g. [22].

We will say that the motion is "quasi-equidistant" if $A_k \ll 1$, where

$$A_k = \left(\max(\xi_{k+1} - \xi_k) - \min(\xi_{k+1} - \xi_k) \right) / r_0, \qquad (6)$$

then the solitons will not asymptotically separate, but instead will slowly oscillate with some small amplitude.

¿From here on, we will denote the various sets of initial conditions for the N-soliton trains by quadruplets $N|r_0|\mathbf{A}|\mathbf{Ph}$, where N is the number of pulses, r_0 is the distance between neighboring pulses and $\mathbf{A}$ and $\mathbf{Ph}$ stand for the configuration of initial amplitudes and phases respectively. In our runs, listed in the tables, the pulses are initially equidistant (i.e. $\xi_{k+1,0} - \xi_{k,0} = r_0 = 8$, which corresponds to 4 pulsewidths), and the initial velocities, $2\mu_{k0}$ are all vanishing. We use here several configurations of amplitudes:

$\mathbf{A}_1$: the amplitudes are monotonic, i.e. $\Delta\nu_0 = \nu_{k+1,0} - \nu_{k,0} = 2d_1$ and such that the average amplitude equals $2\nu_0$; this we denote by the number $2\Delta\nu_0.10^2$;

$\mathbf{A}_2$: configurations with two different alternating amplitudes $\nu_{1,0}$, $\nu_{2,0}$, $\nu_{1,0}$, $\ldots$;

$\mathbf{A}_{3(4)}$: configurations with three (or four) different alternating amplitudes $\nu_{1,0}$, $\ldots$, $\nu_{3(4);0}$, $\nu_{1,0}$ $\ldots$; these we denote by "3sw" (or "4sw") assuming that $\nu_{1,0}$ is the smallest one.

III. THE SOLITON BOUND STATES

A. Analytical results

For the sake of brevity we present here some of the analytical results only for $N = 3$; it is possible to extend

these results to any N.

We assume that $\sum_{k=1}^{3} q_k(\tau) = 0$, i.e. the center of mass is fixed at the origin. This is compatible with $\operatorname{tr} L_0 = 0$. The initial soliton parameters determine L_0 through (4) and (5) and the regimes of propagation are determined by the eigenvalues of L_0. In particular, the three solitons form a bound state if the eigenvalues $\zeta_k = i\eta_k$ are purely imaginary.

The explicit analytic solution to the $N = 3$ CTC is well known [19,20]. Choosing the "symmetric" one for which $q_2 \equiv 0$, $q_1 = -q_3$ and $\zeta_k = (2-k)i\eta_1$ we find that it is periodic with period $T_3^{\mp} = \pi/(2\nu_0\eta_1^{\pm})$ where

$$2\eta_1^{\pm} = \sqrt{(\Delta\nu_0)^2 \pm (\Delta\nu_{\mathrm{cr},3})^2} \tag{7}$$

$\Delta\nu_{\mathrm{cr},3} = 2\sqrt{2}\,\epsilon_0$ and $\epsilon_0 = \nu_0 e^{-\nu_0 r_0}$. The choices for the soliton parameters here are $\mathbf{A}_1$ and $\mathbf{Ph} = \{0, \delta_{2,0}, 0\}$, and the signs plus and minus correspond to $\delta_{2,0} = 0$ and $\delta_{2,0} = \pi$ respectively. For this solution we find

$$A_1^{\pm} = A_2^{\pm} = \pm\frac{1}{2\nu_0}\ln\frac{z_0^2 \pm 1}{z_0^2}, \tag{8}$$

where $z_0 = |\Delta\nu_0|/\Delta\nu_{\mathrm{cr},3}$. Note that similar results for $A_1^{\pm}$ hold also for the $N = 2$ case with the only difference being that $\Delta\nu_{\mathrm{cr},3}$ should be replaced by $4\epsilon_0$. The motion will be quasi-equidistant if $A_1^{\pm} \ll 1$. The formulas (8) show that the increase of $|\Delta\nu_0|$ diminishes $A_1^{\pm}$. Another way to diminish A_1 for fixed $\Delta\nu_0$ and r_0, is to increase the average amplitude ν_0.

The singular behavior of A_1^{+} for $\Delta\nu_0 \to 0$ corresponds to a singular solution of the CTC; the numeric solution of the NLS shows that the solitons do not collide, but do come rather close to each other at the values of t where the CTC solution develop singularities. We will come back to this question in Subsection III.B below.

On the other hand the singularity of A_1^{-} for $\Delta\nu_0 = \Delta\nu_{\mathrm{cr},3}$ corresponds to the fact that at this critical point the quasi-equidistant regime switches over into the free motion regime. Indeed, as we will see below, for $\Delta\nu_0 < \Delta\nu_{\mathrm{cr},3}$ we have regime (i) and the distance between the solitons grows infinitely, while for $\Delta\nu_0 > \Delta\nu_{\mathrm{cr},3}$ we get regime (ii) and a possible quasi-equidistant behavior.

Let us now describe the particular choices of the soliton parameters, which lead to the regimes described above. Such analysis must be based on the solution of the characteristic equation for L_0, which for $N = 3$ with $\operatorname{tr} L_0 = 0$ and generic choice for $\mathbf{A}$ and $\mathbf{Ph}$ is:

$$\zeta^3 + \zeta p + q = 0, \tag{9a}$$

$$p = \frac{1}{32}\left(d_1^2 + d_2^2 + d_3^2\right) - a_1^2 - a_2^2, \tag{9b}$$

$$q = \frac{i}{4}\left(a_1^2 d_3 + a_2^2 d_1\right) + \frac{i}{64} d_1 d_2 d_3, \tag{9c}$$

where $d_k = 2(\nu_{k,0} - \nu_0)$, and $a_k^2 = -\epsilon_0 e^{i(\delta_{k+1,0} - \delta_{k,0})}$. Now we can use Cardano formulas to evaluate the roots ζ_k which determine the asymptotic regime predicted by CTC. For brevity and simplicity we limit ourselves to two special choices of the initial amplitudes. Skipping the details, we find that the bound state regime occurs for each of the following choices of the soliton parameters.

Case $\mathbf{A}_1$.

ii.a) $\mathrm{Ph} \equiv \{0, 0, 0\}$ and $|\Delta\nu_0| \geq 0$;

ii.b) $\mathrm{Ph} \equiv \{0, \pm\pi, 0\}$ and $|\Delta\nu_0| > 2\sqrt{2}\epsilon_0 = \Delta\nu_{\mathrm{cr},3}$;

ii.c) $\mathrm{Ph} \equiv \{0, \delta_0, \delta_0\}$ and $\mathrm{Ph} \equiv \{0, 0, \delta_0\}$ where $\delta_0 = \pm\pi$ and $|\Delta\nu_0| > 2 \cdot 3^{3/4}\epsilon_0$;

Case $\mathbf{A}_2$.

ii.d) $\mathrm{Ph} \equiv \{0, 0, 0\}$ and $|\Delta\nu_0| \geq 0$;

ii.e) $\mathrm{Ph} \equiv \{0, \pm\pi, 0\}$ and $|\Delta\nu_0| > 4\sqrt{2}\epsilon_0$;

ii.f) $\mathrm{Ph} \equiv \{0, \delta_{2,0}, 0\}$ and $|\Delta\nu_0| \geq 0$ if $\cos\delta_{2,0} > 0$; if $\cos\delta_{2,0} < 0$ then the bound state regime holds for $|\Delta\nu_0| > 4\sqrt{-2\cos\delta_{2,0}}\epsilon_0$;

As one may expect the asymptotic behavior depends very much on the initial choice of phases. Thus in the cases **ii.a)** and **ii.d)** the bound state regime is obtained for any value of $\Delta\nu_0$, while in all other cases this regime is entered only if $|\Delta\nu_0|$ is larger than some critical value, which of course, depends on the initial parameters, compare e.g. **ii.b), i.c), ii.e)** and **ii.f)**.

B. Comparison Between CTC and NLS

In this subsection we compare the analytical results obtained above with the numerical solutions of the unperturbed NLS ($R[u] = 0$) (1) with initial conditions (2).

In tables I, II and III we list the values of $\max A_k$ obtained from the numerical solution of the NLS equation with the corresponding CTC solution. It is seen that the agreement is very good and improves with the increasing of $\Delta\nu_0$ and r_0. In addition we show that $\max A_k$ (6) indeed has very low values, which is a reason to call such regime quasi-equidistant.

We find an excellent match between CTC and NLS for the description of all soliton parameters in most cases. The exceptions in this respect happen for those sets of soliton parameters at which: a) CTC has singular or degenerate solutions and b) transition from one regime to another takes place. The choice of $\mathbf{A}_1$ with $\Delta\nu_0 = 0$ and $\delta_{k,0} = 0$ leads to singular solution of CTC for any N. On the other hand the NLS solutions are always analytic and never have singularities. Our numerical checks show that such sets of initial conditions correspond either to solitons collisions or soliton coalescence depending on the value of r_0. In the regions where this happens, the adiabatic approximation is no longer valid and the CTC does not match with the numeric solutions. For $N = 3$ and $r_0 = 8$ the first coalescence takes place at $t_1 \simeq (T_3^{+}/2)|_{\Delta\nu_0=0}$ and the next ones tend to repeat periodically with period very close to $T_3^{+}|_{\Delta\nu_0=0}$ (7). The quasi-equidistant propagation of the solitons is maintained for $t \leq 0.9t_1$.

For $2\Delta\nu_0 \geq 0.1$ we see that in the quasi-equidistant regime the soliton positions display periodic behavior with small periods (much less than t_1) and with small amplitudes of oscillations given by A_k. In several occasions we extended the runs to lengths of $t = 600$ and 996 getting the same type of behavior and the same results for A_k. Thus we conclude that these sets of initial parameters lead to a very stable behavior. In particular, for $\mathbf{A}_1$ configurations with $2\Delta\nu_0 = 0.2$ and $r_0 = 8$ we have an equidistant behavior with an error $A_k \lesssim 10\%$.

The bound state regime is characteristic for $\mathbf{A}_1$ trains with any N. However the $\mathbf{A}_1$ configurations become impractical if one needs a very long sequences of bytes all equal to one, so we should explore other possibilities.

One of them is to use a soliton train of in-phase solitons, with $\mathbf{A}_2$ amplitude configurations [2], where $2\nu_1 = 2\nu_3 = 1.0$, $2\nu_2 = 2\nu_4 = 1.25$ and the run length is about 140. This idea was verified experimentally by [3] where 20 Gbit/s single channel soliton transmission over 11 500 km using alternating amplitude solitons was reported. To check and develop this idea we did a series of runs of N in-phase soliton trains with $\mathbf{A}_2$, $\mathbf{A}_3$ and $\mathbf{A}_4$ amplitude configurations with $N = 3$ to 8, see Tables II and III. For the $\mathbf{A}_2$ and $\mathbf{A}_3$ configurations we find that the quasi-equidistant propagation takes place for $t \leq 0.9T_{\text{qed}}$ with $T_{\text{qed}}^{\text{NLS}} \sim 250$. For larger values of t some of the solitons come rather close to each other. CTC also predicts the same type of behavior but with larger value for $T_{\text{qed}}^{\text{CTC}} \sim 460$. This is the reason why in Table II the numerical data from NLS shows non equidistant propagation when considered for run length 300. We extended some of these runs to lengths of 600 and 996. The results for $N = 3$ show a structure close to a periodic one with a period of about $2T_{\text{qed}}^{\text{NLS}}$. So if the goal is to achieve a quasi-equidistant propagation to lengths $t \leq T_{\text{qed}}^{\text{NLS}}$ then such configurations can be used.

For larger run lengths other configurations must be used, see also [1]. The next possibility which we explored is the $\mathbf{A}_4$ configurations of soliton amplitudes. The results are collected in Table III for $N = 4$ to 8. We see a substantially different picture. Both the NLS and CTC show a quasi-equidistant behavior with the same small values of A_k like in the last column of Table III to run lengths of 1 200; for the CTC this behavior persists even to lengths of 12 000.

IV. CONCLUSIONS

A method for the description of the asymptotic behavior of the N-soliton pulse trains of the NLS equation is proposed, based on the CTC model for the soliton interaction. It describes correctly several qualitatively different classes of asymptotic regimes. Several sets of soliton parameters have been described for which the propagation is quasi-equidistant. Such behavior with a conveniently low value for A_k can be achieved by: a) taking soliton trains with $\Delta\nu_0$ large enough; b) increasing the value $2\nu_0$ of the average amplitude; and c) increasing the distance r_0 between the neighboring solitons.

The critical values of the soliton parameters, for which one regime switches over to another one have been evaluated. We find that near these critical values, the match between CTC and NLS becomes worse, as expected.

The CTC-model provides one with a tool for constructing sets of initial data for the N-soliton trains which will possess a given asymptotic trait, determined by the eigenvalues ζ_k of L_0.

The monotonically increasing amplitudes cannot be used in case one needs trains with larger number of solitons. That is why we investigated $\mathbf{A}_2$, $\mathbf{A}_3$ and $\mathbf{A}_4$ amplitude configurations, for which the propagation is quasi-equidistant. We find that the $\mathbf{A}_2$ and $\mathbf{A}_3$ amplitudes configurations may provide quasi-equidistant propagation to run lengths of the order of $T_{\text{qed}}^{\text{NLS}}$. The $\mathbf{A}_4$ configurations with four different amplitudes show quasi-equidistant behavior to much larger run lengths.

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TABLE I. The value of $\max A_k$ from NLS and CTC for $r_0 = 8$ and $\mathbf{A_1}$ configuration. Run lengths equal 300.

NLS	CTC	NLS	CTC	NLS	CTC
2|8|10|0		2|8|15|0		2|8|20|0	
0.055	0.054	0.029	0.026	0.019	0.016
3|8|10|0		3|8|15|0		3|8|20|0	
0.099	0.03	0.053	0.014	0.031	0.008
4|8|10|0		4|8|15|0		4|8|20|0	
0.14	0.13	0.085	0.051	0.064	0.026
5|8|10|0		5|8|15|0		5|8|20|0	
0.20	0.09	0.13	0.026	0.11	0.019

TABLE II. The value of $\max A_k$ from NLS and CTC for $r_0 = 8$ and $\mathbf{A_2}$ configuration: by "sl" we have denoted trains with $2\nu_1 = 1.0$ and $2\nu_2 = 1.25$ and "ls" means $2\nu_1 = 1.25$ and $2\nu_2 = 1.00$. Run lengths equal 300.

NLS	CTC	NLS	CTC	NLS	CTC
3|8|sl|0		3|8|ls|0		4|8|sl|0	
0.62	0.08	0.012	0.026	0.42	0.078
4|8|ls|0		5|8|sl|0		5|8|ls|0	
0.42	0.078	0.23	0.058	0.59	0.031
6|8|sl|0		6|8|ls|0		7|8|sl|0	
0.27	0.039	0.28	0.039	0.26	0.073
7|8|ls|0		8|8|sl|0		8|8|ls|0	
0.27	0.035	0.25	0.04	0.31	0.04

TABLE III. The value of $\max A_k$ from NLS and CTC for $r_0 = 8$ and $\mathbf{A_3}$, and $\mathbf{A_4}$ configurations: 3sw $\to \{0.85, 1.00, 1.15, 0.85, \ldots\}$ and 4sw $\to \{0.85, 1.00, 1.15, 1.30, 0.85, \ldots\}$. Run lengths equal 1200.

NLS	CTC	NLS	CTC	NLS	CTC
4|8|3sw|0		5|8|3sw|0		5|8|4sw|0	
0.19	0.07	0.47	0.07	0.06	0.04
6|8|3sw|0		6|8|4sw|0		7|8|3sw|0	
0.46	0.20	0.10	0.04	0.79	0.44
7|8|4sw|0		8|8|3sw|0		8|8|4sw|0	
0.09	0.04	0.84	0.33	0.09	0.03